NATIONAL OPEN UNIVERSITY OF NIGERIA

University Village, Plot 91, Cadastral Zone, Nnamdi Azikiwe Expressway, Jabi, Abuja

FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS

COURSE CODE: MTH402
COURSE TITLE: GENERAL TOPOLOGY II
TIME ALLOWED: 3 Hours
TOTAL MARKS: 70 Marks
Instruction: Answer Question One and Any other Four Questions

(1a) Define
(i) A Topological space **(2 Marks)**
(ii) Hausdorff space **(2 Marks)**
(iii) Dense sets in a topological space **(2 Marks)**

(b) Let X be a nonempty set and τ be a collection of subsets of X. Prove that the intersection $\tau = \bigcap_{\alpha\in\Delta} \tau_\alpha \; (\alpha \in \Delta, \; for \; some \; index \; set \; \Delta)$ on X is itself a topology on X. **(4 Marks)**

(c)(i) Let X be a Hausdorff space. Prove that for all $x \in X$, the singleton set $\{x\}$ is closed.

(4 Marks)

(ii) Let X be a Hausdorff space. Prove that a sequence of points of X converges to at most one point of X, that is if a sequence $\{x_n\}$ in X, converges, the limit is unique. **(4 Marks)**

(d) Let (X, τ) be a topological space and let A be a subset of X. Prove that A is dense if and only if every non-empty open subset U of X implies $U \cap A = \phi$. **(4 Marks)**

(2a) Define a separation of a topological space. **(2 Marks)**

(b)(i) Prove that the image of a connected topological space under a continuous function is connected. **(2 Marks)**

(ii) Prove a finite Cartesian product of connected spaces is connected. **(2 Marks)**

(c)(i) Define a Lesbesgue number of an open cover of a topological space. **(2 Marks)**

(ii) State and prove the extremum value theorem of topological spaces. **(4 Marks)**

(3a) Define

(i) Baire topological space. **(2 Marks)**

(ii) Metrizable topological space **(2 Marks)**

(iii) Neighbourhood basis **(2 Marks)**

(b) Prove the sequence Lemma of topological spaces:

(i) If X is a topological space and A is a subset of X, if there exists a sequence $\{x_n\}$ of elements converging to $x \in X$, *then* $x \in \overline{A}$ that the converse holds if X is metrizable. **(3 Marks)**

(ii) If X and Y are topological spaces and $f: X \to Y$ is a function. If the function X is continuous, then for every sequence $\{x_n\}$ *in* X such that $\{x_n\}$ converges to a point x in X, then the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y and that the converse is true if X is metrizable. **(3 Marks)**

(4a)(i) Define a basis for a topology X. **(3 Marks)**

(b) Let B and B* be a basis for the topologies τ *and* τ^* respectively on X. Prove that the following are equivalent:

(i) τ^* is finer than τ. **(3 Marks)**

(ii) For each $x \in X$ and each basis element $B_1 \in B$ containing x, there exists a basis element $B_2 \in B^*$ such that $B_2 \subset B.$ **(3 Marks)**

(b) Let d be a metric on the set X. Prove that the collection of all balls $B_\delta(x, \delta)$ *for* $x \in X$ *and* $\delta > 0$ is a basis for a topology on X, called the metric topology induced by δ. **(3 Marks)**

(5a)(i) Define a subspace topology **(2 Marks)**

(5a)(ii) Prove that the collection $S = \{\pi_1^{-1}(U); U \text{ is open in } X\} \cup \{\pi_2^{-1}(V): V \text{ is open in } Y\}$ is a sub-basis for the product topology on $X \times Y$. **(4 Marks)**

(b)(i) Define the neighbourhood of the element of topological space **(2 Marks)**

(ii) Let A be a subset of the topological space X. Let A^* be the set of all limit points of A. Prove that $\overline{A} = A \cup A^*$. **(4 Marks)**

(6a)(i) Define a path in a topological space. Hence, when is a topological space said to be path connected? **(2 Marks)**

(ii) Prove that the connected components of a topological space X are connected disjoint subspaces of X, whose union is X, such that each nonempty connected subspace of X intersects only one of them. **(4 Marks)** (b)(i) Define the interior and closure of the subset of a topological space X. **(3 Marks)**

(ii) Let Y be a subspace of a topological space X. Prove that asset A is closed in Y if and only if it is equal to the intersection of a closed set of X with Y. **(3 Marks)**