NATIONAL OPEN UNIVERSITY OF NIGERIA
University Village Plot 91, Cadastral Zone, Nnamdi Azikiwe Expressway, Jabi, Abuja

FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
2021_1 Examinations

Course Code:	**MTH301**
Course Title:	**Functional Analysis I**
Credit Unit:	**3**
Time Allowed:	**3 Hours**
Total:	**70 Marks**
Instruction:	**Answer Question One (1) and Any Other 4 Questions**

(1) (a) Define (i) a complete metric space **(2 Marks)**
(ii) a first category set **(2 Marks)**
(iii) a Countable set **(2 Marks)**
(iv) a Open ball. **(2 Marks)**

(b) (i) Let X be a complete metric space and $\{O_n\}$ be a countable collection of dense open subsets of X. Prove that that the union $\cup O_n$ is not empty. **(5 Marks)**

(ii) Prove that in $\mathfrak{R}^n$, every family of disjoint non-empty open set is countable. **(3 Marks)**

(c) (i) Prove that in $\mathfrak{R}^n$, the union of arbitrary collection of open sets is open. **(3 Marks)**

(ii) Prove that the finite intersection of a collection of open sets is open. **(3 Marks)**

(2) (a) Define the followings
(i) a Metric space **(2Marks)**
(ii) Pseudometric **(2 Marks)**
(iii) Distance between two vectors **(2 Marks)**

(b) Explain the concept of an ordered field. **(6 Marks)**

(3) (a) Define the followings:
(i) a function **(2 Marks)**
(ii) Real-valued function **(2 Marks)**
(iii) Vector-valued function. **(2 Marks)**

(b) Let A and B be metric spaces. Prove that $f: A \to B$ is continuous if and only if $f^{-1}(V)$ is an open set in A whenever V is open in B. **(6 Marks)**

(4) (a) Define the following:

(i)Totally bounded metric space **(2 Marks)**

(ii) Sequentially compact metric space **(2 Marks)**

(b) Let X be a metric space and let Y be a subspace of X. Prove that

(i) If X is compact and Y is closed in X, then Y is compact. **(4 Marks)**

(ii) If Y is compact, then it is closed in X. **(4 Marks)**

(5) a(i) Define a topological space **(2 Marks)**

(ii) Give three examples of topological spaces **(3 Marks each)**

(b) Let (K, d_K) be a compact metric space. Let (Y, d_Y) be any metric space and let $f: K \rightarrow Y$ be continuous. Prove that f(K) is compact. **(4 Marks)**

(6) a(i) If (A, d_A) *and* $(B,\ d_B)$ are metric spaces, when is a function

$f: A \rightarrow B$ said to be continuous? **(2 Marks)**

a(ii) Let f and g be real-valued functions with domain(f) = Range (g) = D ⊂ $\mathcal{R}^N$. If $\lim_{x\to x_0} f(x) = l \ and \ \lim_{x\to x_0} g(x) = m$ then state the three limit theorems concerning the sum, product and quotient of the above function. **(2 Marks each)**

(b) Given the function $f: \Re^2 \rightarrow \Re \ and \ (x_0, y_0) = (1,3)$. Compute the limits of the following functions as $(x, y) \rightarrow (1,3)$:

(i) $\lim_{(x_0,y_0)\to(1,3)} f(x, y), if\ f(x, y) = \frac{2x}{x^2+y^{2+1}}$. **(2 Marks)**

(ii) $\lim_{(x_0,y_0)\to(1,3)} f(x, y), if\ f(x, y) = x^2 + y^2 + 1.$ **(2 Marks)**