NATIONAL OPEN UNIVERSITY OF NIGERIA
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FACULTY OF SCIENCES
April Examination 2019

Course Code:	**MTH312**
Course Title:	**Abstract Algebra II**
Credit Unit:	**3**
Time Allowed:	**3 HOURS**
Total:	**70 Marks**
Instruction:	**ATTEMPT NUMBER ONE (1) AND ANY OTHER FOUR (4) QUESTIONS**

1. (a) Define a quotient group. **(2marks)**

(b) Let H be a subgroup of a group G. Prove that the following equivalent:
(i). H is normal in G.
(ii). $g^{-1}Hg \subseteq H \ \forall\, g \in G$.
(iii). $g^{-1}Hg = H \ \forall\, g \in G$.

(9marks)

(c) Let S be a subring of a ring R. Show that the inclusion map defined by $i: S \to R$ defined by $i(x) = x$ is a homomorphism. Hence, obtain Ker f and Im f.

(5marks)

(d) (i). Find the kernel of the homomorphism $f: Z \to Z_{12}$ defined by $f(x) = \bar{x}$
(ii). What are the ideals of Z_{12}.

(6marks)

2. (a). Define a ring for a non-empty set R. **(2marks)**

(b) Consider the set $Z + iZ = \{m + in: \ m \ and \ n$ are integers$\}$, where $i^2 = -1$.

Verify that $Z + iZ$ is a ring under addition and multiplication of complex number.

(5marks)

(c) Let X be a non-empty set, $\wp(X)$ be the collection of all subsets of $X \ and \ \Delta$ denote the symmetric difference operation. Show that $(\wp(X), \ \Delta, \ \cap)$ is a ring.

(5marks)

3. (a) Define the following:
(i). External direct product
(ii) Internal direct product **(2marks)**

(b) Let a group G be internal direct product of its subgroups H and K. Prove that:

each $x \in G$ can be uniquely expressed as $x = hk$, where $h \in H, k \in K$;

$hk = kh \; \forall \; h \in H, k \in K$. **(5marks)**

(c) (i). Show that a group G order of 255 $(3 \times 5 \times 17)$ has either 1 or 51 Sylow 5-subgroups.

(ii). How many Sylow 3-subgroups can it have? **(5marks)**

4. (a) (i) Define a group homomorphism.

Let $f: G_1 \to G_2$ be a homomorphism. Define:

(ii). image of f.

(iii). kernel of f. **(3marks)**

(b) Consider the groups $(R, +)$ and $(C, +)$ and define $f: (C, +) \to (R, +)$ by $f(x + iy) = x$, the real part of $x + iy$. Show that f is a homomorphism. Hence, find the Im f and Ker f **(5marks)**

(c) If $f: G_1 \to G_2$ and $g: G_2 \to G_3$ are two group homomorphisms, show that the composite map $g \circ f: G_1 \to G_3$ is also a group homomorphism. **(4marks)**

5. (a) (i). When is a permutation $f \in S_n$ said to be r −cyclic?

(ii). Prove that every permutation in S_n $(n \geq 2)$ can be written as a product of transpositions **(4marks)**

(b) (i). Show that $(S_n, \circ)$ is a non-commutative group for $n \geq 3$.

(ii). Do the cycles $(1\ 3)$ and $(1\ 5\ 4)$ commute? Give reason for your answer. **(6marks)**

(c) Express the following cycles as products of transpositions:

(i). $(1\ 3\ 5)$ (ii). $(5\ 3\ 1)$ (iii). $(2\ 4\ 5\ 3)$. **(2marks)**

6. (a) Define the following in relation to groups:

(i). Isomorphism

(ii). Automorphism. **(2marks)**

(b) (i). If $\emptyset: G \to H$ and $\theta: H \to K$ are two isomorphisms of groups, then show that $\theta \circ \emptyset$ is an isomorphism of G onto K.

(ii). Prove that any cyclic group is isomorphic to $(Z, +)$ or $(Z_n, +)$. **(6marks)**

(c) Obtain the image of fg $\in Inn\ G$, where

(i). $G = GL_2(R)$ and $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(ii). $(G, X) = (Z, X)$ and $g = 3$

(iii). $G = Z/5Z$ and $g = 4$. **(4marks)**